7.

1) $a_1 + a_7 = 42$
$a_{10} - a_3 = 21$

$a_1 + a_1 + 6d = 42$
$a_1 + 9d - a_1 - 2d = 21$

$2a_1 + 6d = 42$
$7d = 21 \Rightarrow \boxed{d = 3}$

$2a_1 + 6 \cdot 3 = 42$
$2a_1 = 42 - 18$
$a_1 = \frac{24}{2}$
$\boxed{a_1 = 12}$

2) $a_5 + a_{11} = -0,2$
$a_4 + a_{10} = 2,6$

$a_1 + 4d + a_1 + 10d = -92$
$a_1 + 3d + a_1 + 9d = 2,6$

$2a_1 + 14d = -92$
$2a_1 + 12d = 2,6$ $\}-$

$2d = -2,8 \;/:2$
$d = -1,4$

$2a_1 + 12 \cdot (-1,4) = 2,6$
$2a_1 + (-16,8) = 2,6$
$2a_1 = 2,6 + 16,8$
$2a_1 = 19,4 \;/:2$
$a_1 = 9,7$

3) $a_1 + a_5 = 24$
$a_2 \cdot a_3 = 60$

$a_1 + a_1 + 4d = 24$
$(a_1 + d)(a_1 + 2d) = 60$

$2a_1 + 4d = 24 \;/:2$
$a_1 + 2d = 12$
$a_1 = 12 - 2d$

$(12 - 2d + d)(12 - 2d + 2d) = 60$
$12(12 - d) = 60 \;/:12$
$12 - d = 5$
$12 - 5 = d$
$\boxed{d = 7}$

$a_1 = 12 - 2 \cdot 7 = 12 - 14$
$\boxed{a_1 = -2}$

4) $a_2 + a_3 + a_4 = 3$
$a_1 \cdot a_4 = -20$

$a_1 + d + a_1 + 2d + a_1 + 3d = 3$
$a_1 \cdot (a_1 + 3d) = -20$

$3a_1 + 6d = 3 \;/:3$
$a_1 + 2d = 1$
$a_1 = 1 - 2d$

$(1 - 2d)(1 - 2d + 3d) = -20$
$(1 - 2d)(1 + d) = -20$
$1 + d - 2d - 2d^2 + 20 = 0$
$-2d^2 - d + 21 = 0 \;/(-1)$
$2d^2 + d - 21 = 0$
$d_{1/2} = \frac{-1 \pm \sqrt{1 + 168}}{4}$
$d_{1/2} = \frac{-1 \pm 13}{4}$
$d_1 = \frac{-14}{4} = -\frac{7}{2}$
$d_2 = \frac{12}{4} = 3$

$(a_1)_1 = 1 + 2 \cdot \frac{7}{2} = 8$
$(a_1)_2 = 1 - 2 \cdot 3 = -5$

$\boxed{a_1 = 8 \;\; d = -\frac{7}{2}}$ $\boxed{a_1 = -5 \;\; d = 3}$

5) $a_2 + a_3 + a_4 + a_5 = 34$

$a_2 \cdot a_3 = 28$

$a_1 + d + a_1 + 2d + a_1 + 3d + a_1 + 4d = 34$

$4a_1 + 10d = 34 \;/:2$

$2a_1 + 5d = 17$

$2a_1 = 17 - 5d$

$a_1 = \frac{17-5d}{2}$

$(a_1 + d)(a_1 + 2d) = 28$

$\left(\frac{17-5d}{2} + d\right)\left(\frac{17-5d}{2} + 2d\right) = 28$

$\frac{17-5d+2d}{2} \cdot \frac{17-5d+4d}{2} = 28 \;/\cdot 4$

$(17 - 3d)(17 - d) = 112$

$289 - 17d - 51d + 3d^2 = 112$

$3d^2 - 68d + 177 = 0$

$d_{1/2} = \frac{68 \pm \sqrt{4624 - 2124}}{6}$

$d_{1/2} = \frac{68 \pm 50}{6}$ $\quad d_1 = \frac{118}{6} = \frac{59}{3}$ $\quad d_2 = 3$

$(a_1)_1 = \frac{17 - 5 \cdot \frac{59}{3}}{2} = \frac{\frac{51-295}{3}}{2} = -\frac{244}{6} = -\frac{122}{3}$

$(a_1)_2 = \frac{17 - 5 \cdot 3}{2} = \frac{17-15}{2} = 1$

$\boxed{\begin{matrix} a_1 = -\frac{122}{3} \\ d = \frac{59}{3} \end{matrix}}$ $\quad \boxed{\begin{matrix} a_1 = 1 \\ d = 3 \end{matrix}}$

6) $a_5^2 + a_7^2 = 122$

$a_1 + a_7 = 4$

$(a_1 + 2d)^2 + (a_1 + 6d)^2 = 122$

$a_1 + a_1 + 6d = 4$

$2a_1 = 4 - 6d \;/:2$

$a_1 = 2 - 3d$

$(2 - 3d + 2d)^2 + (2 - 3d + 6d)^2 = 122$

$(2 - d)^2 + (2 + 3d)^2 = 122$

$4 - 4d + d^2 + 4 + 12d + 9d^2 - 122 = 0$

$10d^2 + 8d - 114 = 0 \;/:2$

$5d^2 + 4d - 57 = 0$

$d_{1/2} = \frac{-4 \pm \sqrt{16 + 1140}}{10} = \frac{-4 \pm 34}{10} \Rightarrow$ $d_1 = \frac{-38}{10} = -3{,}8$ $\quad d_2 = \frac{+30}{10} = 3$

$(a_1)_1 = 2 - 3 \cdot (-3{,}8) = 2 + 11{,}4 = 13{,}4$

$(a_1)_2 = 2 - 3 \cdot 3 = 2 - 9 = -7$

$\boxed{\begin{matrix} a_1 = 13{,}4 \\ d = -3{,}8 \end{matrix}}$ $\quad \boxed{\begin{matrix} a_1 = -7 \\ d = 3 \end{matrix}}$

8.

$a_1 = 51$

$d = 48 - 51$

$d = -3$

$a_{19} = 51 - 3 \cdot 18$

$= 51 - 54$

$\boxed{a_{19} = -3}$

prvi član niza s negativnim predznakom

$a_n < 0$

$a_1 + (n-1)d < 0$

$51 + (n-1) \cdot (-3) < 0$

$51 - (n-1) \cdot 3 < 0$

$-(n-1) \cdot 3 < -51 \quad / : (-3)$

$n - 1 > 17$

$n > 18$

9.

$a_1 = -3$

$d = -\frac{44}{15} + 3$

$d = \frac{-44 + 45}{15}$

$d = +\frac{1}{15}$

$a_{47} = a_1 + 46d$

$a_{47} = -3 + \frac{46}{15}$

$\boxed{a_{47} = +\frac{1}{15}}$ prvi pozitivni član niza

$a_n > 0$

$a_1 + (n-1)d > 0$

$-3 + (n-1) \cdot \frac{1}{15} > 0 \quad / \cdot 15$

$-45 + n - 1 > 0$

$n > 45 + 1$

$n > 46$

10.

$\sqrt{x}, \sqrt{5x+4}, \sqrt{12x+13}$

Za članove ovih triju niza vrijedi:

$2a_n = a_{n-1} + a_{n+1}$

$2\sqrt{5x+4} = \sqrt{x} + \sqrt{12x+13} \quad / ^2$

$4(5x+4) = x + 2\sqrt{x}\sqrt{12x+13} + 12x+13$

$20x + 16 = x + 2\sqrt{x(12x+13)} + 12x + 13$

$7x + 3 = 2\sqrt{x(12x+13)} \quad / ^2$

$49x^2 + 42x + 9 = 4x(12x+13)$

$49x^2 + 42x + 9 = 4x(12x+13)$

$49x^2 + 42x + 9 - 48x^2 - 52x = 0$

$x^2 - 10x + 9 = 0$

$x_{1,2} = \frac{10 \pm \sqrt{100 - 36}}{2} = \frac{10 \pm 8}{2} \Rightarrow \frac{x_1 = 9}{x_2 = 1}$ ✓

Uvjeti: $\boxed{x > 0}$

$5x + 4 > 0 \Rightarrow x > -\frac{4}{5}$

$12x + 13 > 0 \Rightarrow x > -\frac{13}{12}$

11. $\frac{a_{n-1}}{\sqrt{x+1}}, \frac{a_n}{\sqrt{5x+9}}, \frac{a_{n+1}}{\sqrt{12x+25}}$

Za aritmetički niz vrijedi $2a_n = a_{n-1} + a_{n+1}$

$2\cdot\sqrt{5x+9} = \sqrt{x+1} + \sqrt{12x+25} \quad /^2$

$4(5x+9) = x+1 + 2\sqrt{12x^2+25x+12x+25} + 12x+25$

$20x+36 = 13x+26 + 2\sqrt{12x^2+37x+25}$

$7x+10 = 2\sqrt{12x^2+37x+25} \quad /^2$

$49x^2+140x+100 = 4(12x^2+37x+25)$

$49x^2+140x+100 = 48x^2+148x+100$

$x^2-8x=0$

$x(x-8)=0$

$x_1=0$

$x-8=0$

$x_2=8$

Uvjeti:

$x+1\geq 0 \quad 5x+9\geq 0$

$x\geq -1 \quad x\geq -\frac{9}{5}$

$12x+25\geq 0$

$x\geq -\frac{25}{12}$

12. $\alpha, \beta, \gamma \Rightarrow \alpha = \beta - d$, $\gamma = \beta + d$ — ako su članovi aritmetičkog niza

$$\frac{\sin\alpha - \sin\gamma}{\cos\gamma - \cos\alpha} = \operatorname{ctg}\beta$$

$$\frac{\sin(\beta-d) - \sin(\beta+d)}{\cos(\beta+d) - \cos(\beta-d)} =$$

$$= \frac{-2\cos\frac{\beta-d+\beta+d}{2}\sin\frac{\beta-d-\beta-d}{2}}{-2\sin\frac{\beta+d+\beta-d}{2}\sin\frac{\beta+d-\beta+d}{2}} =$$

$$= \frac{\cos\beta\sin(-d)}{-\sin\beta\sin d} = \frac{-\cos\beta\sin d}{-\sin\beta\sin d} = \operatorname{ctg}\beta$$

36. $a_1+a_3+a_5+a_7+a_9+a_{11}=72$. Kako je $\frac{a_1+a_{11}}{2}=\frac{a_3+a_9}{2}=\frac{a_5+a_7}{2}=a_6$, to je $6\cdot a_6=72 \Rightarrow a_6=12$. Sada je $a_1+a_6+a_{11}=a_6+(a_1+a_{11})=a_6+2a_6=36$.

37. $\frac{n}{2}[2\cdot 64+(n-1)\cdot(-2)]=750$

$64n-n(n-1)=750 \Rightarrow n^2-65n+750=0 \Rightarrow n_1=15;\ n_2=50$

38. $a_2+a_4+\dots+a_{2n}=126$. Kako je $\frac{a_2+a_{2n}}{2}=\frac{a_4+a_{2n-2}}{2}=\dots=21$, a takvih parova u zadanom zbroju ima n, onda je $21\cdot n=126 \Rightarrow n=6$.

39. Neka je broj članova niza $n=2k$. Tada je

$(a_2+a_4+\dots+a_{2k})-(a_1+a_3+\dots+a_{2k-1})=6$, odnosno,

$(a_2-a_1)+(a_4-a_3)+\dots+(a_{2k}-a_{2k-1})=6 \Rightarrow d+d+\dots+d=6$ tj. $k\cdot d=6$

Kako je $a_{2k}-a_1=10,5 \Rightarrow (2k-1)d=10,5 \Rightarrow 2kd-d=10,5$ tj. $12-d=10,5$

pa je $d=1,5$.

Sada je $k\cdot 1,5=6 \Rightarrow k=4 \Rightarrow n=2k=8$ članova.

Zadaci pod brojevima 40, 41, 42, 43, 44, 45 i 46
rješeni su u rješenjima zbirke.

47. $S_9=\frac{9}{2}(a_1+a_9)=\frac{9}{2}[(a_3-2d)+(a_7+2d)]=\frac{9}{2}(a_3+a_7)=\frac{9}{5}\cdot 10=45$

48. Stoviše $a_1=1, a_2=3, a_3=5, \dots \Rightarrow a_n=2n-1$

$b_1=4, b_2=7, b_3=10, \dots \Rightarrow b_n=3n+1$

Sada je $1+3+5+\dots+(2n-1)=S_n \Rightarrow S_n=\frac{n}{2}(a_1+a_n)=\frac{n}{2}(1+2n-1)=n^2$ i

$4+7+10+\dots+(3n-1)=S_n' \Rightarrow S_n'=\frac{n}{2}(b_1+b_n)=\frac{n}{2}(4+3n+1)=\frac{(3n+5)n}{2}$

Stoga je $\frac{n^2}{\frac{(3n+5)n}{2}}=\frac{40}{7n} \Rightarrow \frac{2n}{3n+5}=\frac{40}{7n} \Rightarrow 7n^2-60n-100=0 \Rightarrow n=10$

49. $5^2\cdot 5^4\cdot 5^6\cdot\dots\cdot 5^{2x}=(0,04)^{-28}$

$5^{2+4+6+\dots+2x}=5^{2\cdot(-28)}$

$2+4+6+\dots+2x=56 \Rightarrow 1+2+3+\dots+x=28 \Rightarrow \frac{x(x+1)}{2}=28$, odnosno,

$x^2+x-56=0 \Rightarrow x=7$

55.

$$S_n = n a_1 + \frac{n(n-1)}{2} d, \quad a_1 = n, \quad d = 2$$

$$28 = n^2 + \frac{n(n-1)}{2} \cdot 2$$

$$28 = n^2 + n^2 - n$$

$$2n^2 - n - 28 = 0$$

$$n_{1,2} = \frac{+1 \pm \sqrt{1 + 224}}{4} = \frac{1 \pm 15}{4}$$

$$n_1 = 4$$

$$n_2 = \frac{-7}{2} \Rightarrow \text{nije prirodan, jer } n \in \mathbb{N}$$

$$\boxed{\begin{aligned} n &= 4 \\ a_1 &= 4 \\ d &= 2 \end{aligned}}$$

56.

$$a_1 + a_2 + a_3 = S_3 = 15$$

$$a_{n-2} + a_{n-1} + a_n = 78$$

$$S_n = 155$$

$$S_{n-3} = 155 - 78$$

$$S_3 = 15$$

$$S_n = n a_1 + \frac{n(n-1)}{2} d$$

$$3a_1 + \frac{3(3-1)}{2} d = 15 \quad (1)$$

$$(n-3)a_1 + \frac{(n-4)(n-3)}{2} d = 77 \quad (2)$$

$$n a_1 + \frac{n(n-1)}{2} d = 155 \quad (3)$$

$$3a_1 + 3d = 15 \;/:3 \quad (1)$$

$$a_1 + d = 5$$

$$\boxed{a_1 = 5 - d}$$

uvrstimo $a_1 = 5 - d$ u (2) i (3)

$$(n-3)(5-d) + \frac{n^2 - 3n - 4n + 12}{2} \cdot d = 77$$

$$n(5-d) + \frac{n^2 - n}{2} d = 155$$

$$5n - nd - 15 + 5d + \frac{n^2 - 7n + 12}{2} \cdot d = 77 \;/\cdot 2$$

$$5n - nd + \frac{n^2 - n}{2} d = 155 \;/\cdot 2$$

$$10n - 2nd - 30 + 6d + n^2 d - 7nd + 12d = 154$$

$$10n - 2nd + n^2 d - nd = 310$$

$$n^2 d - 9nd + 18d + 10n - 30 = 154 \quad (*)$$

$$n^2 d - 3nd + 10n = 310 \quad (4)$$

$$d(n^2 - 3n) = 310 - 10n \quad (4)$$

$$d = \frac{310 - 10n}{n^2 - 3n} \Rightarrow \text{uvrstimo u } (*)$$

59. $A_n = \frac{10}{2}[2\cdot n + (40-1)\cdot(2n-1)] = 20(2n + 78n - 39) = 20(80n-39)$

$A_1 + A_2 + \dots + A_{10} = \frac{10}{2}\left[2\cdot[20(80\cdot 1 - 39)] + (10-1)\cdot[20(-80\cdot 1 + 39 + 80\cdot 2 - 39)]\right] =$

$= 5[2\cdot 820 + 9\cdot 1600] = 80200$ //

60. $a_n < 0 \Rightarrow 1{,}5n < 48 \Rightarrow n < 32 \Rightarrow n = 31$

$a_1 = 1{,}5 - 48 = -46{,}5 \quad ; \quad d = 1{,}5$

$S_{31} = \frac{31}{2}[2\cdot(-46{,}5) + 30\cdot 1{,}5] = -744$ //

61. $a_{n+1} = a_n + 1 \Rightarrow d = 1$, a kako $a_1 = 2$, to je $S_{100} = \frac{100}{2}[2\cdot 2 + 99\cdot 1] = 5150$ //

62. $\left.\begin{aligned} a_m &= a_1 + (m-1)d = \frac{1}{m} \\ a_n &= a_1 + (n-1)d = \frac{1}{n} \end{aligned}\right\} \Rightarrow /\text{oduzimamo}/ : (m-n)d = \frac{1}{m} - \frac{1}{n} = \frac{n-m}{mn} \Rightarrow d = \frac{1}{mn}$

$a_1 + \frac{n-1}{mn} = \frac{1}{m} \;/\cdot m \Rightarrow a_1 = \frac{1}{mn}$

$S_{mn} = \frac{mn}{2}\left[2\cdot\frac{1}{mn} + (mn-1)\frac{1}{mn}\right] = \frac{mn}{2}\cdot\frac{2+mn-1}{mn} = \frac{mn+1}{2}$ //

63. $\left.\begin{aligned} a_m &= a_1 + (m-1)d = n \\ a_n &= a_1 + (n-1)d = m \end{aligned}\right\} /\text{oduzmemo}/ : (m-n)d = n - m \Rightarrow d = -1$

$a_1 + (m-1)\cdot(-1) = n \Rightarrow a_1 = n + m - 1$

$S_{m+n} = \frac{m+n}{2}\left[2(n+m-1) + (m+n-1)\cdot(-1)\right] = \frac{m+n}{2}[n+m-1]$ //

64. $a_x = \frac{1}{8}S_{x-1}$. Kako je $a_1 = 20$ i $d = -2$, to je $a_x = 20 + (x-1)\cdot(-2) = 22 - 2x$, i

$S_{x-1} = \frac{x-1}{2}[2\cdot 20 + (x-2)\cdot(-2)] = -x^2 + 23x - 22$.

Tako je $22 - 2x = \frac{1}{8}(-x^2 + 23x - 22) \Rightarrow x^2 - 39x + 198 = 0 \Rightarrow x_1 = 6;\ x_2 = 33$,

pa su to članovi a_6 i a_{33} //.

65. $S_n = \frac{1}{9}S_{3n} \Rightarrow \frac{n}{2}[2a_1 + (n-1)d] = \frac{1}{9}\cdot\frac{3n}{2}[2a_1 + (3n-1)d] \;/\cdot\frac{6}{n}$

$6a_1 + 3d(n-1) = 2a_1 + (3n-1)d \Rightarrow 4a_1 = 2d \;/:2 \Rightarrow d = 2a_1$

$\Rightarrow$ niz $a_1, 3a_1, 5a_1, 7a_1, \dots$ //